Instructor: Girish Varma • Course Code: IMA410 5 problems • 5 marks each

1 Intersecting Families

1. Let $n = 2k$. Show that there are $2^{\binom{2k-1}{k-1}}$ intersecting families of k element subsets of $[n]$ having the maximum number $\binom{2k-1}{k-1}$ of members. (3 marks)

2. Show that for every non-empty subset A of $[n]$, there is an intersecting family $\mathcal{F}$ of subsets of $[n]$ of size 2^{n-1} with $A \in \mathcal{F}$. Show further that any two subsets A, B with $A \cap B \neq \emptyset$ are contained in a family with these properties. What about three pairwise intersecting sets?

   (2 marks)

2 System of Distinct Representatives

1. How many different systems of distinct representatives are there for the sets $\{A_1, \cdots, A_n\}$ where $A_i = [n] \setminus \{i\}$ where $n \geq 2$? (3 marks)

2. Let $A_1, A_2, \cdots, A_n$ be subsets of $\{1, 2, \cdots, n\}$. Let M be the $n \times n$ matrix whose (i, j)th entry is 1 if $j \in A_i$ and 0 otherwise. Prove that the number of SDRs of $(A_1, A_2, ..., A_n)$ is at least $|\det(M)|$ (the determinant). (2 marks)

3 Rings, Fields and Vector Spaces

Consider the n dimensional vector space $\mathbb{F}_p^n$ over the field $\mathbb{F}_p$ (p is a prime).

1. Count the number of distinct tuples of vectors $(v_1, \cdots, v_r)$ such that v_i's are linearly independent (need to justify). (2 marks)

2. Count the number of distinct 1 dimensional sub spaces (need to justify). (1 mark)

3. Count the number of distinct r dimensional sub spaces (need to justify). (2 marks)

P.T.O.

4 Inclusion Exclusion

1. Let $k \geq r$. Show that: (2 marks)

   (a) For odd r,
   $$\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^r \binom{k}{r} \leq 0.$$

   (b) For even r,
   $$\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^r \binom{k}{r} \geq 0.$$

2. Let $A_1, ..., A_k \subseteq [n]$. For $S \subseteq [k]$, let $A_S = \cap_{i \in S} A_i$ and $A_\phi = [n]$. Then for $0 \leq r \leq k$, show that (using part 1.) (3 marks)

   (a) For odd r,
   $$\left|\overline{\cup_{i \in k} A_k}\right| \geq \sum_{S \subseteq [k] : |S| \leq r} (-1)^{|S|} |A_S|.$$

   (b) For even r,
   $$\left|\overline{\cup_{i \in k} A_k}\right| \leq \sum_{S \subseteq [k] : |S| \leq r} (-1)^{|S|} |A_S|.$$

5 Pólya - Burnside Counting

Let k be a prime and $\mathcal{F} = \{f : [m]^k \to [n]\}$ (set of all functions from $[m]^k$ to $[n]$). g is a cyclic reordering of f, if $\exists i \in [k]$ such that $g(x) = f(\sigma^i(x))$ where $\sigma^i(x) = x_i \cdots x_n x_1 \cdots x_{i-1}$.

1. Find the number of distinct functions, which are invariant under cyclic reordering. That is find (2 marks)
   $$\left|\left\{ f \in \mathcal{F} : \forall i \in [k], \forall x \in [m]^k, f(x) = f(\sigma^i(x)) \right\}\right|.$$

2. Find the number of distinct functions in $\mathcal{F}$ if cyclic reordering are considered the same. (2 marks)

3. Show that k divides $n^{m^k} - n^t$ where $t = \frac{m^k - m}{k} + m$. (1 mark)