# Advanced Math Structures
# Quiz 1

August 28, 2019

# Directions

- Keep bags, mobiles and other items outside or near the board.
- Find a seat with an answer sheet kept. Write name, roll number at the top of the answer sheet.
- Solve any 2 problems. Stop at 840AM.
- No additional sheets will be given.

**Q1.** A function $f : [m]^k \to [n]$ is symmetric if for all $x, y \in [m]^k$ such that $y$ can be obtained by permuting $x$ (using a permutation in $S_k$), $f(x) = f(y)$. Find the number of symmetric functions in terms of $m, n, k$.

**Q2.** Let $n \geq 2$ be a positive integer with prime factorization $n = p_1^{e_1}.p_2^{e_2}\ldots p_k^{e_k}$. The Euler totient function is defined by

$\phi(n) = |\{m : m < n \text{ and } \gcd(m, n) = 1\}|$.

Show using inclusion exclusion that

$$\phi(n) = n\prod_{i=1}^{k}\left(\frac{p_i - 1}{p_i}\right).$$

**Q3.** Let $k$ be prime and $\mathcal{F} = \{f : [m]^k \to [n]\}$. $g$ is a cyclic reordering of $f$, if $\exists i \in [k]$ such that $g(x) = f(\sigma^i(x))$ where $\sigma^i(x) = x_i \cdots x_n x_1 \cdots x_{i-1}$. Find the number of distinct functions in $\mathcal{F}$ if cyclic reorderings are considered the same. Use this to show that $k$ divides $n^{m^k} - n^t$ where $t = \frac{m^k - m}{k} + m$.

**Q4.** Let $A_1, \ldots, A_k \subseteq [n]$. For $S \subseteq [k]$, let $A_S = \cap_{i \in S} A_i$ and $A_\phi = [n]$. Then for $0 \leq r \leq k$, show that

1. For odd $r$,
$|\overline{\cup_{i \in k} A_k}| \geq \sum_{S \subseteq [k] : |S| \leq r} (-1)^{|S|} |A_S|$.
2. For even $r$,
$|\overline{\cup_{i \in k} A_k}| \leq \sum_{S \subseteq [k] : |S| \leq r} (-1)^{|S|} |A_S|$.

Thanks