Instructor: Girish Varma • Course Code: MA5.401• IIIT Hyderabad Solve 3 of 4 problems

1 Ising Model on Graphs

Consider a directed graph $G(V \cup \{s,t\}, E)$ (V is the vertex set with $|V| = n$ and E is the edge set). For every vertex $i \in V$, associate a spin random variable X_i taking values in $\{-1,+1\}$. The energy of a configuration $x = (x_1, \cdots, x_n)$ is given by

$$H(x) = -\sum_{(i,j)\in E} x_i x_j - \sum_{(s,i)\in E} x_i + \sum_{(i,t)\in E} x_i.$$

a.) For $n = 3$ and the graph being a chain (ie. $E = \{(1,2),(2,3),(s,1),(3,t)\}$), write down the probability of the configuration $(+1,+1,-1)$ under the Boltzmann's Distribution at temperature $T = 10$. **(1)**

b.) Let $y = (-1,+1,-1,+1,\cdots,(-1)^n)$. What is the probability of the configuration y in the Boltzmann's distribution as $T \to \infty$? **(1)**

c.) Consider the graph with the directed edges given by $E = \{(i,j) : i < j \text{ where } i,j \in \{1,\cdots,n\}\} \cup \{(s,1),(n,t)\}$. Describe the Boltzmann's distribution as $T \to 0$? **(1.5)**

d.) Derive that the ground states (maximum probability states) are given by the Minimum $s-t$ cut in the graph. **(1.5)**

2 MCMC Sampling

Consider an undirected graph $G(V,E)$. Consider random variables X_i for $i \in V$ (one each for every vertex) taking values in $\{1,\cdots n+1\}$ with the potential function (of the Markov Network) being

$$p(x) \propto \prod_{(i,j)\in E} \phi(x_i,x_j) \prod_{i\in V} x_i \qquad \text{where} \qquad \phi(a,b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{otherwise} \end{cases}.$$

a.) Describe the highest and 0 probability states in the distribution where the graph is a 3×3 grid (9 variables). **(1)**

b.) Describe a Markov Chain (MC) with states space $\Omega = \{1,\cdots,n+1\}^{|V|}$, with transitions between states $x,y \in \Omega$ only possible if they differ in at most 1 coordinate such that there is a path from any x to any y. **(1.5)** c.) What should be the transition probabilities such that the stationery distribution of the chain is the distribution described above? (need to give transition probabilities, derive the stationery distribution) **(2.5)**

3 Tail Bounds

Suppose we throw m balls into n bins (uniformly and independently). Balls $\{i, j\}$ is said to *collide* if they fall into the same bin. Let $X_{m,n}$ be the random variable corresponding to the number of collisions and $\mu_{m,n}$ be its expected value.

a.) Show that $\mu_{m,n} = \binom{m}{2}\frac{1}{n}$. **(1)**

b.) Using Chebyshev's inequality show that

$$\Pr[|X_{m,n} - \mu_{m,n}| \geq c\sqrt{\mu_{m,n}}] \leq \frac{1}{c^2}.$$

(2)

c.) Let $m < \sqrt{n}$. Use Chernoff's bounds plus the union bound to show that the probability that no bin has more than 1 ball is at least $1 - n \cdot 2 \cdot e^{-m/8}$. **(2)**

4 Message Passing

Consider the distribution given by

$$p(v_1, \cdots, v_T, h_1, \cdots, h_T) = p(h_1)p(v_1 \mid h_1)\prod_{i=2}^{T} p(v_i \mid h_i)p(h_i \mid h_{i-1})$$

where the domains of h_i's is $\{1, \cdots, H\}$ and v_i's is $\{1, \cdots, V\}$.

a.) Draw Belief Network for the above distribution. **(1)**

b.) Draw factor graph representation for the above distribution. **(1)**

c.) Use the factor graph and message passing to obtain an algorithm with running time $O(TH)$ for computing $p(h_1 \mid v_1, \cdots v_T)$. **(1.5)**

d.) Use the factor graph and message passing to obtain an algorithm with running time $O(T(H+V))$ for computing $p(h_1 \mid v_T)$. **(1.5)**
